

Part 4 : "New Advanced Study of Magic Squares and Cubes"

Chapter 4 : Commentary Articles No.2:

"Various Arts and Tools for Studying Magic Squares"

Section 1 : Notations and 'Euler Squares': Kanji Setsuda

#0. Modern Methods for Studying Magic Objects

In this section let me explain a little about some **notations** expressed by the positional writing system of numbers and about the concept of "Euler Squares".

They are really the important topics of modern methods for studying magic things. Without having any knowledge of them, you might not get any master key to open the secret doors inviting you to the wonderful world of magic squares and cubes.

#1. 'Classical' Notation and 'Mathematical'

There are various forms of expressing our concept of magic squares and cubes. I want you to see some examples listed below.

The one named 'Classical' Notation is one of the most familiar forms of all. It uses a series of natural numbers beginning with '1' for the numerical elements of our object. Of course, they are expressed in the conventional number system, 'Decimal'.

The next form named 'Mathematical' Notation uses the serial integers beginning with '0'. It is made after the classical one, always subtracting 1 from the value of each element. Many modern researchers prefer to use this style for their basic forms, since it is more suitable to deal with the new number systems than the decimal one.

/Classic	/Math.-----	/N2i	/D2i
1 12 7 14	6 13 0 11 6 13 0 11	0000 1011 0110 1101	0101 0011 0110 0101
8 13 2 11	1 10 7 12 1 10 7 12	0111 1100 0001 1010	0101 1100 1001 1010
10 3 16 5	15 4 9 2 15 4 9 2	1001 0010 1111 0100	1010 0011 0110 1010
15 6 9 4	8 3 14 5 8 3 14 5	1110 0101 1000 0011	1010 1100 1001 0101
	-----		*8/ *4/ *2/ *1/

The diagram '/N2i' in the center shows the new style of notation expressing every number in the positional number system of base 2, so called 'Binary Number System'.

The last '/D2i' diagram is made after the /N2i taking each value-part according to its position and placing it in the separate layers denoting their own positions: $2^3/$, $2^2/$, $2^1/$ and $2^0/$. I am afraid of my simple explanations confusing you, but I recommend you to make them actually by yourself, know them very well, and get used to them.

#2. What is the Positional Writing System of Numbers?

I don't intend to give you any educational lessons here to teach about the positional number system. I only want to explain about what is related to our magic square world. If you want to know more about it, you should ask to your own math teachers.

New number systems limit their usable numerical letters to the count of base N.

For instance, Decimal number system uses only ten numerical letters: {0, 1, 2, 3, 4, 5, 6, 7, 8 and 9} to express all values. You can express ten by '10' with the two letters '1' and '0'. We all know '10' means $1 \times 10 + 0 = 10$ _[Dec], and '25' means $2 \times 10 + 5 = 25$ _[Dec].

Positional number system of the Base 9 uses only nine numerical letters: {0, 1, 2, 3, 4, 5, 6, 7 and 8} to express all values. You can express nine by '10'. We know '10_[N9i]' means $1 \times 9 + 0 = 9$ _[Dec], '25_[N9i]' means $2 \times 9 + 5 = 23$ _[Dec], and '67_[N9i]' means $6 \times 9 + 7 = 61$ _[Dec].

Positional number system of the Base 7 uses only seven numerical letters: {0, 1, 2, 3, 4, 5 and 6} to express all values. You can express seven by '10'. We know '10_[N7i]' means $1 \times 7 + 0$

=7_[Dec], '25_[N7i]' means $2 \times 7 + 5 = 19$ _[Dec], and '63_[N7i]' means $6 \times 7 + 3 = 45$ _[Dec].

Positional number system of the Base 5 uses only five numerical letters: {0, 1, 2, 3, and 4} to express all values. You can express five by '10'. We know '10_[N5i]' means $1 \times 5 + 0 = 5$ _[Dec], '43_[N5i]' means $4 \times 5 + 3 = 23$ _[Dec], and '213_[N5i]' means $2 \times 25 + 1 \times 5 + 3 = 58$ _[Dec].

Positional number system of the Base 3 uses only three numerical letters: {0, 1 and 2} to express all values. You can express three by '10'. We know '10_[N3i]' means $1 \times 3 + 0 = 3$ _[Dec]. '21_[N3i]' means $2 \times 3 + 1 = 7$ _[Dec], and '2010_[N3i]' means $2 \times 27 + 0 \times 9 + 1 \times 3 + 0 = 57$ _[Dec].

Binary number system (of the Base 2) uses only two numerical letters: {0 and 1} to express all values. You can express two by '10'. We know '10_[N2i]' means $1 \times 2 + 0 = 2$ _[Dec]. '101_[N2i]' means $1 \times 4 + 0 \times 2 + 1 = 5$ _[Dec], and '11010_[N2i]' means $1 \times 16 + 1 \times 8 + 0 \times 4 + 1 \times 2 + 0 = 26$ _[Dec].

**** Correspondence among Various Notations ****

CIs	Mat	N10i	N9i	N8i	N7i	N6i	N5i	N4i	N3i	N2i/
1	0	0	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1	1	1	1
3	2	2	2	2	2	2	2	2	2	10
4	3	3	3	3	3	3	3	3	10	11
5	4	4	4	4	4	4	4	10	11	100
6	5	5	5	5	5	5	10	11	12	101
7	6	6	6	6	10	11	12	20	20	110
8	7	7	7	7	10	11	12	13	21	111
9	8	8	8	10	11	12	13	20	22	1000
10	9	9	10	11	12	13	14	21	100	1001
11	10	10	11	12	13	14	20	22	101	1010
12	11	11	12	13	14	15	21	23	102	1011
13	12	12	13	14	15	20	22	30	110	1100
14	13	13	14	15	16	21	23	31	111	1101
15	14	14	15	16	20	22	24	32	112	1110
16	15	15	16	17	21	23	30	33	120	1111
17	16	16	17	20	22	24	31	100	121	10000
18	17	17	18	21	23	25	32	101	122	10001
19	18	18	20	22	24	30	33	102	200	10010
20	19	19	21	23	25	31	34	103	201	10011
21	20	20	22	24	26	32	40	110	202	10100
22	21	21	23	25	30	33	41	111	210	10101
23	22	22	24	26	31	34	42	112	211	10110
24	23	23	25	27	32	35	43	113	212	10111
25	24	24	26	30	33	40	44	120	220	11000
26	25	25	27	31	34	41	100	121	221	11001
27	26	26	28	32	35	42	101	122	222	11010
28	27	27	30	33	36	43	102	123	1000	11011
29	28	28	31	34	40	44	103	130	1001	11100
30	29	29	32	35	41	45	104	131	1002	11101
31	30	30	33	36	42	50	110	132	1010	11110
32	31	31	34	37	43	51	111	133	1011	11111
33	32	32	35	40	44	52	112	200	1012	100000
34	33	33	36	41	45	53	113	201	1020	100001
35	34	34	37	42	46	54	114	202	1021	100010
36	35	35	38	43	50	55	120	203	1022	100011
37	36	36	40	44	51	100	121	210	1100	100100
38	37	37	41	45	52	101	122	211	1101	100101
39	38	38	42	46	53	102	123	212	1102	100110
40	39	39	43	47	54	103	124	213	1110	100111
41	40	40	44	50	55	104	130	220	1111	101000
42	41	41	45	51	56	105	131	221	1112	101001
43	42	42	46	52	60	110	132	222	1120	101010
44	43	43	47	53	61	111	133	223	1121	101011

45		44		44	48	54	62	112	134	230	1122	101100
46		45		45	50	55	63	113	140	231	1200	101101
47		46		46	51	56	64	114	141	232	1201	101110
48		47		47	52	57	65	115	142	233	1202	101111
49		48		48	53	60	66	120	143	300	1210	110000
50		49		49	54	61	100	121	144	301	1211	110001
51		50		50	55	62	101	122	200	302	1212	110010
52		51		51	56	63	102	123	201	303	1220	110011
53		52		52	57	64	103	124	202	310	1221	110100
54		53		53	58	65	104	125	203	311	1222	110101
55		54		54	60	66	105	130	204	312	2000	110110
56		55		55	61	67	106	131	210	313	2001	110111
57		56		56	62	70	110	132	211	320	2002	111000
58		57		57	63	71	111	133	212	321	2010	111001
59		58		58	64	72	112	134	213	322	2011	111010
60		59		59	65	73	113	135	214	323	2012	111011
61		60		60	66	74	114	140	220	330	2020	111100
62		61		61	67	75	115	141	221	331	2021	111101
63		62		62	68	76	116	142	222	332	2022	111110
64		63		63	70	77	120	143	223	333	2100	111111

CIs | Mat | N10i N9i N8i N7i N6i N5i N4i N3i N2i/

They say Legendary **Leonhard Euler**(1707-1783) already knew about this style of notations and used these positional number systems 250 years ago, but it is at the present time of computer age that we really know we all have to master these arts when we are school boys. Since electrical circuits in any computer can distinguish the difference only between 'On' and 'Off', in another notation '1' and '0', any basic software has to deal with the Binary Number System at first.

#3. How do you convert it to the one in another Number System?

You have to use the **serial divisions** by the base N to convert your decimal number to the one in another number system. We call the way as the 'decomposition' by the base N. See the next examples below. If you repeat your divisions by the same base N one after another, you can finally make your answer with all the remainders.

When you want to make any number by the Base N converted back to the decimal ones, you have to use the **serial multiplications** by the Base N, as shown below.

2) 2 <Decomposition of 2 by the Base 2>

$$\begin{array}{r} 2) \ 1 \ \dots \ 0 \\ \hline 0 \ \dots \ 1 \end{array}$$

$$2_{[Dec]} = 10_{[N2i]}$$

<Composition> $[1] \times 2 + [0] = 2_{[Dec]}$
or $[1] \times 2^1 + [0] \times 2^0 = 2_{[Dec]}$

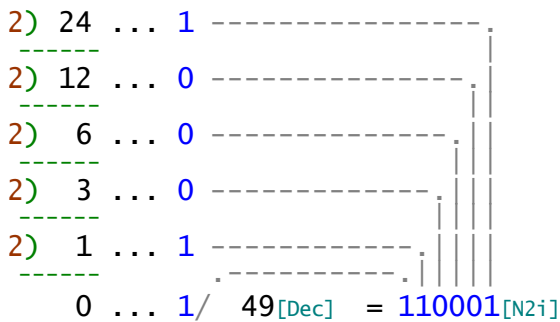
2) 5 <Decomposition of 5 by the Base 2>

$$\begin{array}{r} 2) \ 2 \ \dots \ 1 \\ \hline 2) \ 1 \ \dots \ 0 \\ \hline 0 \ \dots \ 1 \end{array}$$

$$5_{[Dec]} = 101_{[N2i]}$$

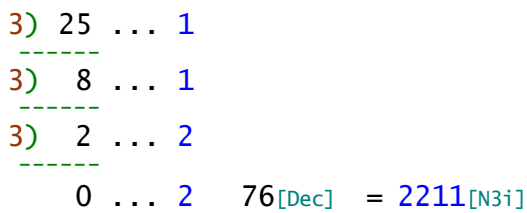
<Composition> $([1] \times 2 + [0]) \times 2 + [1] = 5_{[Dec]}$
or $[1] \times 2^2 + [0] \times 2^1 + [1] \times 2^0 = 5_{[Dec]}$

2) 49 <Decomposition of 49 by the Base 2>



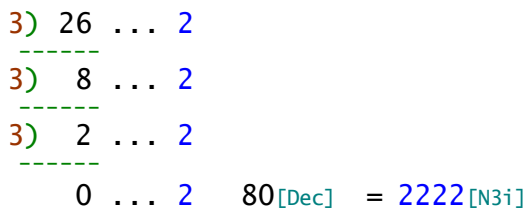
<Composition> $(((((1]x2+[1])x2+[0])x2+[0])x2+[1])=49_{[Dec]}$
 or $[1]x2^5+[1]x2^4+[0]x2^3+[0]x2^2+[0]x2^1+[1]x2^0=49_{[Dec]}$

3) 76 <Decomposition of 76 by the Base 3>



<Composition> $((([2]x3+[2])x3+[1])x3+[1])=76_{[Dec]}$
 or $[2]x3^3+[2]x3^2+[1]x3^1+[1]x3^0=76_{[Dec]}$

3) 80 <Decomposition of 80 by the Base 3>



<Composition> $((([2]x3+[2])x3+[2])x3+[2])=80_{[Dec]}$
 or $[2]x3^3+[2]x3^2+[2]x3^1+[2]x3^0=80_{[Dec]}$

If you want to make your result back into the original decimal number, you may well multiply each position by the base N and add the next position to the product. Repeat the same job one after another until you get to the last position. We call this way as the '**composition**', since it means the reverse conversion of '**decomposition**'.

$$V_{[Dec]} = ((((Ax^N+B)x^N+C)x^N+D)x^N+E)x^N+F;$$

or $V_{[Dec]} = Ax^N^5 + Bx^N^4 + Cx^N^3 + Dx^N^2 + Ex^N^1 + Fx^N^0;$

If you want to have your original values in the Classical Notation, you always have to add 1 to each result in the Mathematical Notation.

#4. Basic Forms of Mathematical Notation and /N5i

Many modern researchers prefer to use such the next forms as below for our magic squares of order 5. The second diagram is called '/N5i' (Notation of the 5th-increment number system). All the 25 numbers are expressed in the positional number system of the base 5, and are placed in the correct locations so that you could see the same value patterns with the x/y co-ordinates of the locations themselves.

Yes. The number patterns are just the same to each other between the values and locations. It is the basis of making some transformations possible, such as the tricky 'Do-it-After-the-Model Transformation' exchanging their information of those two.

<code>/Mathematical</code>	<code>x\y</code>	0	1	2	3	4	Locations in x/y Co-ordinates
0 1 2 3 4	0	00	01	02	03	04	00 _[N5i] is located in (0, 0). 01 _[N5i] is located in (0, 1). 02 _[N5i] is located in (0, 2). 03 _[N5i] is located in (0, 3). 04 _[N5i] is located in (0, 4).
5 6 7 8 9	1	10	11	12	13	14	10 _[N5i] is located in (1, 0). 11 _[N5i] is located in (1, 1).
10 11 12 13 14	2	20	21	22	23	24	20 _[N5i] is located in (2, 0). 21 _[N5i] is located in (2, 1).
15 16 17 18 19	3	30	31	32	33	34	30 _[N5i] is located in (3, 0). 31 _[N5i] is located in (3, 1).
20 21 22 23 24	4	40	41	42	43	44	42 _[N5i] is located in (4, 2). 43 _[N5i] is located in (4, 3). 44 _[N5i] is located in (4, 4).

#5. Basic Forms of Various Notations, `/Nbasei` and `/Dbasei`

I prefer to use all the diagrams listed below for our Basic Forms of magic squares of every order from 3 to 9.

** The Basic Forms with Decomposition Diagrams **

#1 Basic Forms of Order 3

1 2 3	0 1 2	00 01 02	0 0 0	0 1 2
4 5 6	3 4 5	10 11 12	1 1 1	0 1 2
7 8 9	6 7 8	20 21 22	2 2 2	0 1 2
<code>/Classical</code>	<code>/Mathematical</code>	<code>/N3i</code>	<code>/D3i</code>	

#2 Basic Forms of Order 4

1 2 3 4	0 1 2 3	0000 0001 0010 0011	0000 0000 0011 0101
5 6 7 8	4 5 6 7	0100 0101 0110 0111	0000 1111 0011 0101
9 10 11 12	8 9 10 11	1000 1001 1010 1011	1111 0000 0011 0101
13 14 15 16	12 13 14 15	1100 1101 1110 1111	1111 1111 0011 0101
<code>/Classical</code>	<code>/Mathematical</code>	<code>/N2i</code>	<code>/D2i</code>

#3 Basic Forms of Order 5

1 2 3 4 5	0 1 2 3 4	00 01 02 03 04	0 0 0 0 0	0 1 2 3 4
6 7 8 9 10	5 6 7 8 9	10 11 12 13 14	1 1 1 1 1	0 1 2 3 4
11 12 13 14 15	10 11 12 13 14	20 21 22 23 24	2 2 2 2 2	0 1 2 3 4
16 17 18 19 20	15 16 17 18 19	30 31 32 33 34	3 3 3 3 3	0 1 2 3 4
21 22 23 24 25	20 21 22 23 24	40 41 42 43 44	4 4 4 4 4	0 1 2 3 4
<code>/Classical</code>	<code>/Mathematical</code>	<code>/N5i</code>	<code>/D5i</code>	

#4 Basic Forms of Order 6

1 2 3 4 5 6	000000 000001 000010 000011 000100 000101
7 8 9 10 11 12	000110 000111 001000 001001 001010 001011
13 14 15 16 17 18	001100 001101 001110 001111 010000 010001
19 20 21 22 23 24	010010 010011 010100 010101 010110 010111
25 26 27 28 29 30	011000 011001 011010 011011 011100 011101
31 32 33 34 35 36	011110 011111 100000 100001 100010 100011
<code>/Classical</code>	<code>/N2i</code>
0 1 2 3 4 5	000000 000000 000000 000011 001100 010101
6 7 8 9 10 11	000000 000000 001111 110000 110011 010101
12 13 14 15 16 17	000000 000011 111100 111100 001100 010101
18 19 20 21 22 23	000000 111111 000000 001111 110011 010101
24 25 26 27 28 29	000000 111111 111111 000011 001100 010101
30 31 32 33 34 35	001111 110000 110000 110000 110011 010101
<code>/Mathematical</code>	<code>/D2i</code>

#5 Basic Forms of Order 7

1	2	3	4	5	6	7	00	01	02	03	04	05	06
8	9	10	11	12	13	14	10	11	12	13	14	15	16
15	16	17	18	19	20	21	20	21	22	23	24	25	26
22	23	24	25	26	27	28	30	31	32	33	34	35	36
29	30	31	32	33	34	35	40	41	42	43	44	45	46
36	37	38	39	40	41	42	50	51	52	53	54	55	56
43	44	45	46	47	48	49	60	61	62	63	64	65	66

/Classical

/N7i

0	1	2	3	4	5	6	0	0	0	0	0	0	0	0	1	2	3	4	5	6
7	8	9	10	11	12	13	1	1	1	1	1	1	1	0	1	2	3	4	5	6
14	15	16	17	18	19	20	2	2	2	2	2	2	2	0	1	2	3	4	5	6
21	22	23	24	25	26	27	3	3	3	3	3	3	3	0	1	2	3	4	5	6
28	29	30	31	32	33	34	4	4	4	4	4	4	4	0	1	2	3	4	5	6
35	36	37	38	39	40	41	5	5	5	5	5	5	5	0	1	2	3	4	5	6
42	43	44	45	46	47	48	6	6	6	6	6	6	6	0	1	2	3	4	5	6

/Mathematical

/D7i

#6 Basic Forms of Order 8

1	2	3	4	5	6	7	8	000000	000001	000010	000011	000100	000101	000110	000111
9	10	11	12	13	14	15	16	001000	001001	001010	001011	001100	001101	001110	001111
17	18	19	20	21	22	23	24	010000	010001	010010	010011	010100	010101	010110	010111
25	26	27	28	29	30	31	32	011000	011001	011010	011011	011100	011101	011110	011111
33	34	35	36	37	38	39	40	100000	100001	100010	100011	100100	100101	100110	100111
41	42	43	44	45	46	47	48	101000	101001	101010	101011	101100	101101	101110	101111
49	50	51	52	53	54	55	56	110000	110001	110010	110011	110100	110101	110110	110111
57	58	59	60	61	62	63	64	111000	111001	111010	111011	111100	111101	111110	111111

/Classical

/N2i

0	1	2	3	4	5	6	7	00000000	00000000	00000000	00001111	00110011	01010101
8	9	10	11	12	13	14	15	00000000	00000000	11111111	00001111	00110011	01010101
16	17	18	19	20	21	22	23	00000000	11111111	00000000	00001111	00110011	01010101
24	25	26	27	28	29	30	31	00000000	11111111	11111111	00001111	00110011	01010101
32	33	34	35	36	37	38	39	11111111	00000000	00000000	00001111	00110011	01010101
40	41	42	43	44	45	46	47	11111111	00000000	11111111	00001111	00110011	01010101
48	49	50	51	52	53	54	55	11111111	11111111	00000000	00001111	00110011	01010101
56	57	58	59	60	61	62	63	11111111	11111111	11111111	00001111	00110011	01010101

/Mathematical

/D2i

#7 Basic Forms of Order 9

1	2	3	4	5	6	7	8	9	0000	0001	0002	0010	0011	0012	0020	0021	1022
10	11	12	13	14	15	16	17	18	0100	0101	0102	0110	0111	0112	0120	0121	0122
19	20	21	22	23	24	25	26	27	0200	0201	0202	0210	0211	0212	0220	0221	0222
28	29	30	31	32	33	34	35	36	1000	1001	1002	1010	1011	1012	1020	1021	1022
37	38	39	40	41	42	43	44	45	1100	1101	1102	1110	1111	1112	1120	1121	1122
46	47	48	49	50	51	52	53	54	1200	1201	1202	1210	1211	1212	1220	1221	1222
55	56	57	58	59	60	61	62	63	2000	2001	2002	2010	2011	2012	2020	2021	2022
64	65	66	67	68	69	70	71	72	2100	2101	2102	2110	2111	2112	2120	2121	2122
73	74	75	76	77	78	79	80	81	2200	2201	2202	2210	2211	2212	2220	2221	2222

/Classical

/N3i

0	1	2	3	4	5	6	7	8	000000000	000000000	000111222	012012012
9	10	11	12	13	14	15	16	17	000000000	111111111	000111222	012012012
18	19	20	21	22	23	24	25	26	000000000	222222222	000111222	012012012
27	28	29	30	31	32	33	34	35	111111111	000000000	000111222	012012012
36	37	38	39	40	41	42	43	44	111111111	111111111	000111222	012012012
45	46	47	48	49	50	51	52	53	111111111	222222222	000111222	012012012

```

54 55 56 57 58 59 60 61 62   22222222 00000000 000111222 012012012
63 64 65 66 67 68 69 70 71   22222222 11111111 000111222 012012012
72 73 74 75 76 77 78 79 80   22222222 22222222 000111222 012012012
/Mathematical                      /D3i

```

#6. Further Study with /Dbasei Patterns

Let's study further in various forms actually with some samples of solutions next.

I want you to get used to the 'Decomposed Diagram' of the base N such as shown below. '/D3i' means the Decompositions by the 3rd-increment number system.

It is made after the 'N3i' form, taking each value from /N3i, decomposing it by the base 3 and placing each value-part in the separate layer of /D3i for higher position or lower. /H diagram only collects the value-parts of higher positions, and /L collects the ones of lower only. Each keeps the same location information of its original.

* Magic Squares 3x3 *

/Classic		/Math		-----										
2	7	6	1	6	5	01	20	12	0	2	1	1	0	2
9	5	1	8	4	0	22	11	00	2	1	0	2	1	0
4	3	8	3	2	7	10	02	21	1	0	2	0	2	1
						/N3i		D3i/H			/L			

This /D3i form is quite useful for you to see and study the structure of our magic object. For instance, any kind of line sum is easily calculated in this form as follows.

* Check Line-Sums *

```

1+6+5=(0+2+1)*3+(1+0+2)=12 ... rw1 | 1+8+3=(0+2+1)*3+(1+2+0)=12 ... c11
8+4+0=(2+1+0)*3+(2+1+0)=12 ... rw2 | 6+4+2=(2+1+0)*3+(0+1+2)=12 ... c12
3+2+7=(1+0+2)*3+(0+2+1)=12 ... rw3 | 5+0+7=(1+0+2)*3+(2+0+1)=12 ... c13
1+4+7=(0+1+2)*3+(1+1+1)=12 ... pd1 | 5+4+3=(1+1+1)*3+(2+1+0)=12 ... pb1

```

When you get used to it, you will surely be able to see and calculate those sums quickly by heart, since you only have to catch the same 'number pattern' there.

Line-sums of those kinds must be always equal to the Magic Constant in our object. You can easily check and know whether they are equal to that or not.

*** Decompositions of Various Magic Squares ***

** Pan-diagonal Magic Squares 5x5 Decomposed **

1/Classic					/Math.-----														
1	23	20	14	7	13	6	0	22	19	13	6	0	22						
15	9	2	21	18	20	17	14	8	1	20	17	14	8						
22	16	13	10	4	9	3	21	15	12	9	3	21	15						
8	5	24	17	11	16	10	7	4	23	16	10	7	4						
19	12	6	3	25	2	24	18	11	5	2	24	18	11						

/N5i					/D5i					*5/					*1/				
00	42	34	23	11	0	4	3	2	1	0	2	4	3	1					
24	13	01	40	32	2	1	0	4	3	4	3	1	0	2					
41	30	22	14	03	4	3	2	1	0	1	0	2	4	3					
12	04	43	31	20	1	0	4	3	2	2	4	3	1	0					
33	21	10	02	44	3	2	1	0	4	3	1	0	2	4					

* Check Line-Sums *

```

0+22+19+13+6= 00+42+34+23+11= (0+4+3+2+1)*5+(0+2+4+3+1)=60(Dec) ... rw1
14+8+1+20+17= 24+13+01+40+32= (2+1+0+4+3)*5+(4+3+1+0+2)=60(Dec) ... rw2
21+15+12+9+3= 41+30+22+14+03= (4+3+2+1+0)*5+(1+0+2+4+3)=60(Dec) ... rw3

```

$7+4+23+16+10= (1+0+4+3+2)*5+(2+4+3+1+0)= 60 \dots rw4$
 $18+11+5+2+24= (3+2+1+0+4)*5+(3+1+0+2+4)= 60 \dots rw5$
 $0+14+21+7+18= (0+2+4+1+3)*5+(0+4+1+2+3)= 60 \dots c11$
 $22+8+15+4+11= (4+1+3+0+2)*5+(2+3+0+4+1)= 60 \dots c12$
 $19+1+12+23+5= (3+0+2+4+1)*5+(4+1+2+3+0)= 60 \dots c13$
 $13+20+9+16+2= (2+4+1+3+0)*5+(3+0+4+1+2)= 60 \dots c14$
 $6+17+3+10+24= (1+3+0+2+4)*5+(1+2+3+0+4)= 60 \dots c15$
 $0+8+12+16+24= (0+1+2+3+4)*5+(0+3+2+1+4)= 60 \dots pd1$
 $22+1+9+10+18= (4+0+1+2+3)*5+(2+1+4+0+3)= 60 \dots pd2$
 $19+20+3+7+11= (3+4+0+1+2)*5+(4+0+3+2+1)= 60 \dots pd3$
 $13+17+21+4+5= (2+3+4+0+1)*5+(3+2+1+4+0)= 60 \dots pd4$
 $6+14+15+23+2= (1+2+3+4+0)*5+(1+4+0+3+2)= 60 \dots pd5$
 $0+17+9+23+11= (0+3+1+4+2)*5+(0+2+4+3+1)= 60 \dots pb1$
 $22+14+3+16+5= (4+2+0+3+1)*5+(2+4+3+1+0)= 60 \dots pb2$
 $19+8+21+10+2= (3+1+4+2+0)*5+(4+3+1+0+2)= 60 \dots pb3$
 $13+1+15+7+24= (2+0+3+1+4)*5+(3+1+0+2+4)= 60 \dots pb4$
 $6+20+12+4+18= (1+4+2+0+3)*5+(1+0+2+4+3)= 60 \dots pb5$
 ... This is certainly a Pan-diagonal Magic Square!
 $0+24=24 \dots cp1; 22+2=24 \dots cp2; 19+5=24 \dots cp3; 13+11=24 \dots cp4;$
 $6+18=24 \dots cp5; 14+10=24 \dots cp6; 8+16=24 \dots cp7; 1+23=24 \dots cp8;$
 $20+4=24 \dots cp9; 17+7=24 \dots cp10; 21+3=24 \dots cp11; 15+9=24 \dots cp12;$
 $12+12=24 \dots cp13;$
 ... This is certainly a Self-complementary type!

**** Pan-diagonal Magic Squares 7x7 Decomposed ****

1/Classic	/Mathem	----
1 48 39 30 28 19 10	27 18 9 0 47 38 29 27 18 9 0 47 38	
18 9 7 47 38 29 27	37 28 26 17 8 6 46 37 28 26 17 8 6	
35 26 17 8 6 46 37	5 45 36 34 25 16 7 5 45 36 34 25 16	
45 36 34 25 16 14 5	15 13 4 44 35 33 24 15 13 4 44 35 33	
13 4 44 42 33 24 15	32 23 14 12 3 43 41 32 23 14 12 3 43	
23 21 12 3 43 41 32	42 40 31 22 20 11 2 42 40 31 22 20 11	
40 31 22 20 11 2 49	10 1 48 39 30 21 19 10 1 48 39 30 21	

/N7i	/D7i	*7/
00 65 53 41 36 24 12	0 6 5 4 3 2 1	0 5 3 1 6 4 2
23 11 06 64 52 40 35	2 1 0 6 5 4 3	3 1 6 4 2 0 5
46 34 22 10 05 63 51	4 3 2 1 0 6 5	6 4 2 0 5 3 1
62 50 45 33 21 16 04	6 5 4 3 2 1 0	2 0 5 3 1 6 4
15 03 61 56 44 32 20	1 0 6 5 4 3 2	5 3 1 6 4 2 0
31 26 14 02 60 55 43	3 2 1 0 6 5 4	1 6 4 2 0 5 3
54 42 30 25 13 01 66	5 4 3 2 1 0 6	4 2 0 5 3 1 6

*** Check Line-Sums ***

$0+47+38+29+27+18+9= 00+65+53+41+36+24+12= (0+6+5+4+3+2+1)*7+(0+5+3+1+6+4+2)=168 \dots rw1$
 $17+8+6+46+37+28+26= (2+1+0+6+5+4+3)*7+(3+1+6+4+2+0+5)=168 \dots rw2$
 $34+25+16+7+5+45+36= (4+3+2+1+0+6+5)*7+(6+4+2+0+5+3+1)=168 \dots rw3$
 $44+35+33+24+15+13+4= (6+5+4+3+2+1+0)*7+(2+0+5+3+1+6+4)=168 \dots rw4$
 $12+3+43+41+32+23+14= (1+0+6+5+4+3+2)*7+(5+3+1+6+4+2+0)=168 \dots rw5$
 $22+20+11+2+42+40+31= (3+2+1+0+6+5+4)*7+(1+6+4+2+0+5+3)=168 \dots rw6$
 $39+30+21+19+10+1+48= (5+4+3+2+1+0+6)*7+(4+2+0+5+3+1+6)=168 \dots rw7$
 $0+17+34+44+12+22+39= (0+2+4+6+1+3+5)*7+(0+3+6+2+5+1+4)=168 \dots c11$
 $47+8+25+35+3+20+30= (6+1+3+5+0+2+4)*7+(5+1+4+0+3+6+2)=168 \dots c12$
 $38+6+16+33+43+11+21= (5+0+2+4+6+1+3)*7+(3+6+2+5+1+4+0)=168 \dots c13$
 $29+46+7+24+41+2+19= (4+6+1+3+5+0+2)*7+(1+4+0+3+6+2+5)=168 \dots c14$
 $27+37+5+15+32+42+10= (3+5+0+2+4+6+1)*7+(6+2+5+1+4+0+3)=168 \dots c15$
 $18+28+45+13+23+40+1= (2+4+6+1+3+5+0)*7+(4+0+3+6+2+5+1)=168 \dots c16$
 $9+26+36+4+14+31+48= (1+3+5+0+2+4+6)*7+(2+5+1+4+0+3+6)=168 \dots c17$

$0+8+16+24+32+40+48 = (0+1+2+3+4+5+6) * 7 + (0+1+2+3+4+5+6) = 168 \dots pd1$
 $47+6+7+15+23+31+39 = (6+0+1+2+3+4+5) * 7 + (5+6+0+1+2+3+4) = 168 \dots pd2$
 $38+46+5+13+14+22+30 = (5+6+0+1+2+3+4) * 7 + (3+4+5+6+0+1+2) = 168 \dots pd3$
 $29+37+45+4+12+20+21 = (4+5+6+0+1+2+3) * 7 + (1+2+3+4+5+6+0) = 168 \dots pd4$
 $27+28+36+44+3+11+19 = (3+4+5+6+0+1+2) * 7 + (6+0+1+2+3+4+5) = 168 \dots pd5$
 $18+26+34+35+43+2+10 = (2+3+4+5+6+0+1) * 7 + (4+5+6+0+1+2+3) = 168 \dots pd6$
 $9+17+25+33+41+42+1 = (1+2+3+4+5+6+0) * 7 + (2+3+4+5+6+0+1) = 168 \dots pd7$
 $0+26+45+15+41+11+30 = (0+3+6+2+5+1+4) * 7 + (0+5+3+1+6+4+2) = 168 \dots pb1$
 $47+17+36+13+32+2+21 = (6+2+5+1+4+0+3) * 7 + (5+3+1+6+4+2+0) = 168 \dots pb2$
 $38+8+34+4+23+42+19 = (5+1+4+0+3+6+2) * 7 + (3+1+6+4+2+0+5) = 168 \dots pb3$
 $29+6+25+44+14+40+10 = (4+0+3+6+2+5+1) * 7 + (1+6+4+2+0+5+3) = 168 \dots pb4$
 $27+46+16+35+12+31+1 = (3+6+2+5+1+4+0) * 7 + (6+4+2+0+5+3+1) = 168 \dots pb5$
 $18+37+7+33+3+22+48 = (2+5+1+4+0+3+6) * 7 + (4+2+0+5+3+1+6) = 168 \dots pb6$
 $9+28+5+24+43+20+39 = (1+4+0+3+6+2+5) * 7 + (2+0+5+3+1+6+4) = 168 \dots pb7$

... This is certainly a Pan-diagonal Magic Square!

$0+48=48 \dots cp1;$ $47+1=48 \dots cp2;$ $38+10=48 \dots cp3;$ $29+19=48 \dots cp4;$
 $27+21=48 \dots cp5;$ $18+30=48 \dots cp6;$ $9+39=48 \dots cp7;$ $17+31=48 \dots cp8;$
 $8+40=48 \dots cp9;$ $6+42=48 \dots cp10;$ $46+2=48 \dots cp11;$ $37+11=48 \dots cp12;$
 $28+20=48 \dots cp13;$ $26+22=48 \dots cp14;$ $34+14=48 \dots cp15;$ $25+23=48 \dots cp16;$
 $16+32=48 \dots cp17;$ $7+41=48 \dots cp18;$ $5+43=48 \dots cp19;$ $45+3=48 \dots cp20;$
 $36+12=48 \dots cp21;$ $44+4=48 \dots cp22;$ $35+13=48 \dots cp23;$ $33+15=48 \dots cp24;$
 $24+24=48 \dots cp25;$

... This is certainly a Self-complementary type!

Watch the two examples above and know that all line-sums are equal to the Magic Constant. True. Both of them are really the 'Pan-diagonal Magic Squares', as you see.

Could you notice that each line-sum has the same number pattern as follows?

$$V_{sum5} = (0+1+2+3+4) * 5 + (0+1+2+3+4) = 60 \dots (E5) \text{ in PMS55}$$

$$V_{sum7} = (0+1+2+3+4+5+6) * 7 + (0+1+2+3+4+5+6) = 168 \dots (E7) \text{ in PMS77}$$

Each line of each layer consists of {0, 1, 2, 3 and 4} in PMS55, and similarly {0, 1, 2, 3, 4, 5 and 6} in PMS77, and it uses any number strictly once in any line. It has neither repetition nor drop-off of a certain number, as you see.

You don't have to care about the permutation of those elements, since it could never change its sum at all. You have to care for the only constant combination.

It is amazing that all line-sums could be expressed in the same form: 20 equations obey (E5), and all equations as many as 28 obey (E7) above. How beautiful they are!

But this is not my discovery at all. It has been known well in Europe for a very long time as the characteristic property of "Latin Squares" or "Greco-Latin Squares".

They say it is Legendary **Leonhard Euler** that was so positive as to use this property for the purpose of composing any magic squares in modern forms.

#7. "Euler Squares" and "Complete Euler Squares"

It is a habit we used to call "Euler Square" for what has such properties as follows:

(1) Each line in each layer of decomposition must consist of {0, 1, 2, ..., (N-1)}.

You must use any number strictly once in any line. You must not use any number twice or more often. You must not un-use it in any line, either.

(2) In the mathematical notation, any number of {0, 1, 2, ..., (N²-1)} must be used once. No number must be used twice or more often, nor must be unused.

(3) Each line-sum must be expressed in the same form with the next equation:

$$V_{sumN} = \{0+1+2+\dots+(N-1)\} * N + \{0+1+2+\dots+(N-1)\} = \text{Magic_Constant}$$

You don't have to care about the permutation of {0, 1, 2, ..., (N-1)} here. You only have to care for the only constant combination of {0, 1, 2, ..., (N-1)}.

Certainly. Both PMS55 and PMS77 above are the examples of **Euler Squares**.

I would like to call "**Complete Euler Squares**" for those beautiful magic squares, all of whose pan-diagonals beside basic lines obey the 'Euler's Conditions' above.

The first example of MS33 is one of the 'Euler Squares', but it is not the 'Complete Euler Square', because its pan-diagonals do not always obey the Euler's Conditions.

Legendary Leonhard Euler did not mention about pan-diagonals and he might not have known even about 'Complete Euler Squares', Dr. Mutsumi Suzuki once told me.

In order 5, all 3600 pan-diagonal magic squares are Complete Euler Squares, and all 16 Simultaneous types: both self-complementary and pan-diagonal are also Complete Euler Squares, I certified. It is really wonderful all "Suzuki Squares" are found C.E.S.

In order 7, I found 38102400 Complete Euler Squares among pan-diagonal magic squares, and 3456 C.E.S. among Simultaneous types: both self-complementary and pan-diagonal. I was surprised at the big counts of those, and I was more shocked when I found there exist many 'Non-Euler Squares' among them, far more than CES.

#8. How about the Magic Squares of Even Orders?

Let's express some pan-diagonal magic squares of order 4 and 8 in various forms next and know how many Complete Euler Squares we can find.

First of all, let's make them in the decomposed forms by the base 4 and 8, and then by the base 2, that is, the Binary Number System. What do they look like?

See the next examples below, the results of my recent calculations.

** Pan-diagonal Magic Squares 4x4 Decomposed by the Base 4 **

1/Classic	/Math.-----	/N4i	/D4i
1 8 13 12	12 11 0 7 12 11 0 7	00 13 30 23	0 1 3 2 0 3 0 3
14 11 2 7	1 6 13 10 1 6 13 10	31 22 01 12	3 2 0 1 1 2 1 2
4 5 16 9	15 8 3 4 15 8 3 4	03 10 33 20	0 1 3 2 3 0 3 0
15 10 3 6	2 5 14 9 2 5 14 9	32 21 02 11	3 2 0 1 2 1 2 1
	'-----'		*4/ *1/

* Check Line-Sums and See if they are equal to 30:

- 0+7+12+11=00+13+30+23=(0+1+3+2)*4+(0+3+0+3)=30 ... rw1
- 13+10+1+6=(3+2+0+1)*4+(1+2+1+2)=30 ... rw2
- 3+4+15+8=(0+1+3+2)*4+(3+0+3+0)=30 ... rw3
- 14+9+2+5=(3+2+0+1)*4+(2+1+2+1)=30 ... rw4
- 0+13+3+14=(0+3+0+3)*4+(0+1+3+2)=30 ... c11
- 7+10+4+9=(1+2+1+2)*4+(3+2+0+1)=30 ... c12
- 12+1+15+2=(3+0+3+0)*4+(0+1+3+2)=30 ... c13
- 11+6+8+5=(2+1+2+1)*4+(3+2+0+1)=30 ... c14
- 0+10+15+5=(0+2+3+1)*4+(0+2+3+1)=30 ... pd1
- 7+1+8+14=(1+0+2+3)*4+(3+1+0+2)=30 ... pd2
- 12+6+3+9=(3+1+0+2)*4+(0+2+3+1)=30 ... pd3
- 11+13+4+2=(2+3+1+0)*4+(3+1+0+2)=30 ... pd4
- 0+6+15+9=(0+1+3+2)*4+(0+2+3+1)=30 ... pb1
- 7+13+8+2=(1+3+2+0)*4+(3+1+0+2)=30 ... pb2
- 12+10+3+5=(3+2+0+1)*4+(0+2+3+1)=30 ... pb3
- 11+1+4+14=(2+0+1+3)*4+(3+1+0+2)=30 ... pb4

... This is certainly a Pan-diagonal Magic Square!

- 0+5=5 ... cp1; 7+2=9 ... cp2; 12+9=21 ... cp3; 11+14=25 ... cp4;
- 13+8=21 ... cp5; 10+15=25 ... cp6; 1+4=5 ... cp7; 6+3=9 ... cp8;

2/Classic	/Math.-----	/N4i	/D4i
1 8 11 14	10 13 0 7 10 13 0 7	00 13 22 31	0 1 2 3 0 3 2 1
12 13 2 7	1 6 11 12 1 6 11 12	23 30 01 12	2 3 0 1 3 0 1 2
6 3 16 9	15 8 5 2 15 8 5 2	11 02 33 20	1 0 3 2 1 2 3 0
15 10 5 4	4 3 14 9 4 3 14 9	32 21 10 03	3 2 1 0 2 1 0 3
	'-----'		

3/Classic	/Math.---	/N4i	/D4i
1 12 7 14	6 13 0 11 6 13 0 11	00 23 12 31	0 2 1 3 0 3 2 1
8 13 2 11	1 10 7 12 1 10 7 12	13 30 01 22	1 3 0 2 3 0 1 2
10 3 16 5	15 4 9 2 15 4 9 2	21 02 33 10	2 0 3 1 1 2 3 0
15 6 9 4	8 3 14 5 8 3 14 5	32 11 20 03	3 1 2 0 2 1 0 3

Although all of them are pan-diagonal MS44, all the number patterns in all lines do not always obey the Euler's conditions. You can find such a pattern as {0, 0, 3, 3} or {1, 1, 2, 2} in some lines here and there. They are not our Complete Euler Squares at all.

I could find no C.E.S. among pan-diagonal MS44 by the decompositions of base 4.

How about making pan-diagonal MS44 by the decompositions of base 2?

I want you to see the next examples of the same pan-diagonal solutions carefully.

**** Pan-diagonal Magic Squares 4x4 Decomposed by the Base 2 ****

1/Classic	/Math.---	/N2i	/D2i
1 8 13 12	12 11 0 7 12 11 0 7	0000 0111 1100 1011	0011 0110 0101 0101
14 11 2 7	1 6 13 10 1 6 13 10	1101 1010 0001 0110	1100 1001 0101 1010
4 5 16 9	15 8 3 4 15 8 3 4	0011 0100 1111 1000	0011 0110 1010 1010
15 10 3 6	2 5 14 9 2 5 14 9	1110 1001 0010 0101	1100 1001 1010 0101

*8/ *4/ *2/ *1/

*** Check Line-Sums and See if they are equal to 30:**

- 0+7+12+11=(0+0+1+1)*8+(0+1+1+0)*4+(0+1+0+1)*2+(0+1+0+1)= 30 ... rw1
- 13+10+1+6=(1+1+0+0)*8+(1+0+0+1)*4+(0+1+0+1)*2+(1+0+1+0)= 30 ... rw2
- 3+4+15+8=(0+0+1+1)*8+(0+1+1+0)*4+(1+0+1+0)*2+(1+0+1+0)= 30 ... rw3
- 14+9+2+5=(1+1+0+0)*8+(1+0+0+1)*4+(1+0+1+0)*2+(0+1+0+1)= 30 ... rw4
- 0+13+3+14=(0+1+0+1)*8+(0+1+0+1)*4+(0+0+1+1)*2+(0+1+1+0)= 30 ... c11
- 7+10+4+9=(0+1+0+1)*8+(1+0+1+0)*4+(1+1+0+0)*2+(1+0+0+1)= 30 ... c12
- 12+1+15+2=(1+0+1+0)*8+(1+0+1+0)*4+(0+0+1+1)*2+(0+1+1+0)= 30 ... c13
- 11+6+8+5=(1+0+1+0)*8+(0+1+0+1)*4+(1+1+0+0)*2+(1+0+0+1)= 30 ... c14
- 0+10+15+5=(0+1+1+0)*8+(0+0+1+1)*4+(0+1+1+0)*2+(0+0+1+1)= 30 ... pd1
- 7+1+8+14=(0+0+1+1)*8+(1+0+0+1)*4+(1+0+0+1)*2+(1+1+0+0)= 30 ... pd2
- 12+6+3+9=(1+0+0+1)*8+(1+1+0+0)*4+(0+1+1+0)*2+(0+0+1+1)= 30 ... pd3
- 11+13+4+2=(1+1+0+0)*8+(0+1+1+0)*4+(1+0+0+1)*2+(1+1+0+0)= 30 ... pd4
- 0+6+15+9=(0+0+1+1)*8+(0+1+1+0)*4+(0+1+1+0)*2+(0+0+1+1)= 30 ... pb1
- 7+13+8+2=(0+1+1+0)*8+(1+1+0+0)*4+(1+0+0+1)*2+(1+1+0+0)= 30 ... pb2
- 12+10+3+5=(1+1+0+0)*8+(1+0+0+1)*4+(0+1+1+0)*2+(0+0+1+1)= 30 ... pb3
- 11+1+4+14=(1+0+0+1)*8+(0+0+1+1)*4+(1+0+0+1)*2+(1+1+0+0)= 30 ... pb4

... This is certainly a Pan-diagonal Magic Square!

- 0+5=5 ... cp1; 7+2=9 ... cp2; 12+9=21 ... cp3; 11+14=25 ... cp4;
- 13+8=21 ... cp5; 10+15=25 ... cp6; 1+4=5 ... cp7; 6+3=9 ... cp8;

2/Classic	/Math.---	/N2i	/D2i
1 8 11 14	10 13 0 7 10 13 0 7	0000 0111 1010 1101	0011 0101 0110 0101
12 13 2 7	1 6 11 12 1 6 11 12	1011 1100 0001 0110	1100 0101 1001 1010
6 3 16 9	15 8 5 2 15 8 5 2	0101 0010 1111 1000	0011 1010 0110 1010
15 10 5 4	4 3 14 9 4 3 14 9	1110 1001 0100 0011	1100 1010 1001 0101

*8/ *4/ *2/ *1/

3/Classic	/Math.---	/N2i	/D2i
1 12 7 14	6 13 0 11 6 13 0 11	0000 1011 0110 1101	0101 0011 0110 0101
8 13 2 11	1 10 7 12 1 10 7 12	0111 1100 0001 1010	0101 1100 1001 1010
10 3 16 5	15 4 9 2 15 4 9 2	1001 0010 1111 0100	1010 0011 0110 1010
15 6 9 4	8 3 14 5 8 3 14 5	1110 0101 1000 0011	1010 1100 1001 0101

*8/ *4/ *2/ *1/

Could you notice that all the lines including pan-diagonals have the same number pattern {0, 0, 1, 1} in each layer of decompositions by binary number system?

What a beautiful structure they have! This characteristic property looks so beautiful that it should be noted for the new Euler's conditions, I believe. Let's modify a little about the previous Euler's Conditions as follows:

*** New Euler's Conditions for Order 4 ***

(1) Each line in each layer of $/D2i$ forms must consist of {0, 0, 1, 1}. Number '0' must be used as often as '1' in any line. One must not be used more or less often than the other, nor must be un-used in any line.

(2) In the mathematical notation, any number of {0, 1, 2, ..., 14, 15} must be used once. No number must be used twice or more often, nor must be unused.

(3) Each line-sum must be expressed in the same form with the next equation:

$$V_{sum4} = \{0+0+1+1\} * 8 + \{0+0+1+1\} * 4 + \{0+0+1+1\} * 2 + \{0+0+1+1\} = 30 \dots (E4)$$

You don't have to care about the permutation of {0, 0, 1, 1} here. It never changes the line sum. You only have to care for the only constant combination of {0, 0, 1, 1}.

If you accept these new conditions, you may now call all the examples above as '**Complete Euler Squares**', since you can find the same number pattern even in their pan-diagonals. The more C.E.S. of order 4, the better base of decompositions.

I could find 48 Complete Euler Squares of order 4 among all the pan-diagonal magic squares 4x4. It means all the pan-diagonal MS44 are Complete Euler Squares without any exceptions. There is no 'Non-Euler' type among pan-diagonal solutions of order 4.

Let's examine the case of order 8 next, shall we?

**** 'Composite & Complete' Magic Squares 8x8 with /D8i ****

1/Classic	/Mathemat
1 63 4 62 8 58 5 59	7 57 4 58 0 62 3 61 7 57 4 58 0 62 3 61
56 10 53 11 49 15 52 14	48 14 51 13 55 9 52 10 48 14 51 13 55 9 52 10
25 39 28 38 32 34 29 35	31 33 28 34 24 38 27 37 31 33 28 34 24 38 27 37
48 18 45 19 41 23 44 22	40 22 43 21 47 17 44 18 40 22 43 21 47 17 44 18
57 7 60 6 64 2 61 3	63 1 60 2 56 6 59 5 63 1 60 2 56 6 59 5
16 50 13 51 9 55 12 54	8 54 11 53 15 49 12 50 8 54 11 53 15 49 12 50
33 31 36 30 40 26 37 27	39 25 36 26 32 30 35 29 39 25 36 26 32 30 35 29
24 42 21 43 17 47 20 46	16 46 19 45 23 41 20 42 16 46 19 45 23 41 20 42

/N8i	/D8i	*8/	*1/
00 76 03 75 07 71 04 72	0 7 0 7 0 7 0 7	0 6 3 5 7 1 4 2	
67 11 64 12 60 16 63 15	6 1 6 1 6 1 6 1	7 1 4 2 0 6 3 5	
30 46 33 45 37 41 34 42	3 4 3 4 3 4 3 4	0 6 3 5 7 1 4 2	
57 21 54 22 50 26 53 25	5 2 5 2 5 2 5 2	7 1 4 2 0 6 3 5	
70 06 73 05 77 01 74 02	7 0 7 0 7 0 7 0	0 6 3 5 7 1 4 2	
17 61 14 62 10 66 13 65	1 6 1 6 1 6 1 6	7 1 4 2 0 6 3 5	
40 36 43 35 47 31 44 32	4 3 4 3 4 3 4 3	0 6 3 5 7 1 4 2	
27 51 24 52 20 56 23 55	2 5 2 5 2 5 2 5	7 1 4 2 0 6 3 5	

*** Check Line-Sums ***

0+62+3+61+7+57+4+58=	(0+7+0+7+0+7+0+7)*8+(0+6+3+5+7+1+4+2)= 252	... rw1
55+9+52+10+48+14+51+13=	(6+1+6+1+6+1+6+1)*8+(7+1+4+2+0+6+3+5)= 252	... rw2
24+38+27+37+31+33+28+34=	(3+4+3+4+3+4+3+4)*8+(0+6+3+5+7+1+4+2)= 252	... rw3
47+17+44+18+40+22+43+21=	(5+2+5+2+5+2+5+2)*8+(7+1+4+2+0+6+3+5)= 252	... rw4
56+6+59+5+63+1+60+2=	(7+0+7+0+7+0+7+0)*8+(0+6+3+5+7+1+4+2)= 252	... rw5
15+49+12+50+8+54+11+53=	(1+6+1+6+1+6+1+6)*8+(7+1+4+2+0+6+3+5)= 252	... rw6
32+30+35+29+39+25+36+26=	(4+3+4+3+4+3+4+3)*8+(0+6+3+5+7+1+4+2)= 252	... rw7
23+41+20+42+16+46+19+45=	(2+5+2+5+2+5+2+5)*8+(7+1+4+2+0+6+3+5)= 252	... rw8
0+55+24+47+56+15+32+23=	(0+6+3+5+7+1+4+2)*8+(0+7+0+7+0+7+0+7)= 252	... c11

62+9+38+17+6+49+30+41= (7+1+4+2+0+6+3+5)*8+(6+1+6+1+6+1+6+1)= 252 ... c12
 3+52+27+44+59+12+35+20= (0+6+3+5+7+1+4+2)*8+(3+4+3+4+3+4+3+4)= 252 ... c13
 61+10+37+18+5+50+29+42= (7+1+4+2+0+6+3+5)*8+(5+2+5+2+5+2+5+2)= 252 ... c14
 7+48+31+40+63+8+39+16= (0+6+3+5+7+1+4+2)*8+(7+0+7+0+7+0+7+0)= 252 ... c15
 57+14+33+22+1+54+25+46= (7+1+4+2+0+6+3+5)*8+(1+6+1+6+1+6+1+6)= 252 ... c16
 4+51+28+43+60+11+36+19= (0+6+3+5+7+1+4+2)*8+(4+3+4+3+4+3+4+3)= 252 ... c17
 58+13+34+21+2+53+26+45= (7+1+4+2+0+6+3+5)*8+(2+5+2+5+2+5+2+5)= 252 ... c18
 0+9+27+18+63+54+36+45= (0+1+3+2+7+6+4+5)*8+(0+1+3+2+7+6+4+5)= 252 ... pd1
 62+52+37+40+1+11+26+23= (7+6+4+5+0+1+3+2)*8+(6+4+5+0+1+3+2+7)= 252 ... pd2

4+14+31+18+59+49+32+45= (0+1+3+2+7+6+4+5)*8+(4+6+7+2+3+1+0+5)= 252 ... pb7
 58+51+33+40+5+12+30+23= (7+6+4+5+0+1+3+2)*8+(2+3+1+0+5+4+6+7)= 252 ... pb8

**** The same 'C & C' Magic Squares 8x8 with /D4i ****

/N4i		/D4i	*16/		*4/		*1/
000	332 003 331 013 321 010 322	0 3 0 3 0 3 0 3	0 3 0 3 1 2 1 2	0	2 3 1 3 1 0 2		
	313 021 310 022 300 032 303 031	3 0 3 0 3 0 3 0	1 2 1 2 0 3 0 3		3 1 0 2 0 2 3 1		
	120 212 123 211 133 201 130 202	1 2 1 2 1 2 1 2	2 1 2 1 3 0 3 0		0 2 3 1 3 1 0 2		
	233 101 230 102 220 112 223 111	2 1 2 1 2 1 2 1	3 0 3 0 2 1 2 1		3 1 0 2 0 2 3 1		
	320 012 323 011 333 001 330 002	3 0 3 0 3 0 3 0	2 1 2 1 3 0 3 0		0 2 3 1 3 1 0 2		
	033 301 030 302 020 312 023 311	0 3 0 3 0 3 0 3	3 0 3 0 2 1 2 1		3 1 0 2 0 2 3 1		
	200 132 203 131 213 121 210 122	2 1 2 1 2 1 2 1	0 3 0 3 1 2 1 2		0 2 3 1 3 1 0 2		
	113 221 110 222 100 232 103 231	1 2 1 2 1 2 1 2	1 2 1 2 0 3 0 3		3 1 0 2 0 2 3 1		

*** Check Line-Sums ***

0+62+3+61+7+57+4+58= (0+3+0+3+0+3+0+3)*16+(0+3+0+3+1+2+1+2)*4+(0+2+3+1+3+1+0+2)=252 ... rw1
 55+9+52+10+48+14+51+13= (3+0+3+0+3+0+3+0)*16+(1+2+1+2+0+3+0+3)*4+(3+1+0+2+0+2+3+1)=252 ... rw2
 24+38+27+37+31+33+28+34= (1+2+1+2+1+2+1+2)*16+(2+1+2+1+3+0+3+0)*4+(0+2+3+1+3+1+0+2)=252 ... rw3

0+55+24+47+56+15+32+23= (0+3+1+2+3+0+2+1)*16+(0+1+2+3+2+3+0+1)*4+(0+3+0+3+0+3+0+3)=252 ... c11
 62+9+38+17+6+49+30+41= (3+0+2+1+0+3+1+2)*16+(3+2+1+0+1+0+3+2)*4+(2+1+2+1+2+1+2+1)=252 ... c12
 3+52+27+44+59+12+35+20= (0+3+1+2+3+0+2+1)*16+(0+1+2+3+2+3+0+1)*4+(3+0+3+0+3+0+3+0)=252 ... c13

0+9+27+18+63+54+36+45= (0+0+1+1+3+3+2+2)*16+(0+2+2+0+3+1+1+3)*4+(0+1+3+2+3+2+0+1)=252 ... pd1
 62+52+37+40+1+11+26+23= (3+3+2+2+0+0+1+1)*16+(3+1+1+2+0+2+2+1)*4+(2+0+1+0+1+3+2+3)=252 ... pd2
 3+10+31+22+60+53+32+41= (0+0+1+1+3+3+2+2)*16+(0+2+3+1+3+1+0+2)*4+(3+2+3+2+0+1+0+1)=252 ... pd3

4+14+31+18+59+49+32+45= (0+0+1+1+3+3+2+2)*16+(1+3+3+0+2+0+0+3)*4+(0+2+3+2+3+1+0+1)=252 ... pb7
 58+51+33+40+5+12+30+23= (3+3+2+2+0+0+1+1)*16+(2+0+0+2+1+3+3+1)*4+(2+3+1+0+1+0+2+3)=252 ... pb8

**** The same 'C & C' Magic Squares 8x8 with /D2i ****

/N2i		/D2i*32/	*16/	*8/	*4/	*2/	*1/
000000	111110 000011 111101 000111 111001 000100 111010	01010101	01010101	01010101	01011010	01101001	00111100
	110111 001001 110100 001010 110000 001110 110011 001101	10101010	10101010	01010101	10100101	10010110	11000011
	011000 100110 011011 100101 011111 100001 011100 100010	01010101	10101010	10101010	01011010	01101001	00111100
	101111 010001 101100 010010 101000 010110 101011 010101	10101010	01010101	10101010	10100101	10010110	11000011
	111000 000110 111011 000101 111111 000001 111100 000010	10101010	10101010	10101010	01011010	01101001	00111100
	001111 110001 001100 110010 001000 110110 001011 110101	01010101	10101010	10101010	10100101	10010110	11000011
	100000 011110 100011 011101 100111 011001 100100 011010	10101010	01010101	01010101	01011010	01101001	00111100
	010111 101001 010100 101010 010000 101110 010011 101101	01010101	10101010	10101010	10100101	10010110	11000011

*** Check Line-Sums ***

$$\begin{aligned}
0+62+3+61+7+57+4+58 &= (0+1+0+1+0+1+0+1)*32+(0+1+0+1+0+1+0+1)*16+(0+1+0+1+0+1+0+1)*8+ \\
&\quad (0+1+0+1+1+0+1+0)*4+(0+1+1+0+1+0+0+1)*2+(0+0+1+1+1+1+0+0)= 252 \dots \text{rw1} \\
55+9+52+10+48+14+51+13 &= (1+0+1+0+1+0+1+0)*32+(1+0+1+0+1+0+1+0)*16+(0+1+0+1+0+1+0+1)*8+ \\
&\quad (1+0+1+0+0+1+0+1)*4+(1+0+0+1+0+1+1+0)*2+(1+1+0+0+0+0+1+1)= 252 \dots \text{rw2} \\
24+38+27+37+31+33+28+34 &= (0+1+0+1+0+1+0+1+0+1)*32+(1+0+1+0+1+0+1+0+1+0)*16+(1+0+1+0+1+0+1+0)*8+ \\
&\quad (0+1+0+1+1+0+1+0)*4+(0+1+1+0+1+0+0+1)*2+(0+0+1+1+1+1+0+0)= 252 \dots \text{rw3} \\
47+17+44+18+40+22+43+21 &= (1+0+1+0+1+0+1+0)*32+(0+1+0+1+0+1+0+1)*16+(1+0+1+0+1+0+1+0)*8+ \\
&\quad (1+0+1+0+0+1+0+1)*4+(1+0+0+1+0+1+1+0)*2+(1+1+0+0+0+0+1+1)= 252 \dots \text{rw4} \\
56+6+59+5+63+1+60+2 &= (1+0+1+0+1+0+1+0)*32+(1+0+1+0+1+0+1+0)*16+(1+0+1+0+1+0+1+0)*8+ \\
&\quad (0+1+0+1+1+0+1+0)*4+(0+1+1+0+1+0+0+1)*2+(0+0+1+1+1+1+0+0)= 252 \dots \text{rw5}
\end{aligned}$$

$$\begin{aligned}
0+55+24+47+56+15+32+23 &= (0+1+0+1+1+0+1+0+1+0)*32+(0+1+1+0+1+0+0+1)*16+(0+0+1+1+1+1+0+0)*8+ \\
&\quad (0+1+0+1+0+1+0+1)*4+(0+1+0+1+0+1+0+1)*2+(0+1+0+1+0+1+0+1)= 252 \dots \text{c11} \\
62+9+38+17+6+49+30+41 &= (1+0+1+0+0+1+0+1+0)*32+(1+0+0+1+0+1+1+0)*16+(1+1+0+0+0+0+1+1)*8+ \\
&\quad (1+0+1+0+0+1+0+1+0)*4+(1+0+1+0+1+0+1+0+1)*2+(0+1+0+1+0+1+0+1)= 252 \dots \text{c12} \\
3+52+27+44+59+12+35+20 &= (0+1+0+1+1+0+1+0+1+0)*32+(0+1+1+0+1+0+0+1)*16+(0+0+1+1+1+1+0+0)*8+ \\
&\quad (0+1+0+1+0+1+0+1)*4+(1+0+1+0+1+0+1+0)*2+(1+0+1+0+1+0+1+0)= 252 \dots \text{c13} \\
61+10+37+18+5+50+29+42 &= (1+0+1+0+0+1+0+1+0)*32+(1+0+0+1+0+1+1+0)*16+(1+1+0+0+0+0+1+1)*8+ \\
&\quad (1+0+1+0+1+0+1+0+1)*4+(0+1+0+1+0+1+0+1)*2+(1+0+1+0+1+0+1+0)= 252 \dots \text{c14} \\
7+48+31+40+63+8+39+16 &= (0+1+0+1+1+0+1+0+1+0)*32+(0+1+1+0+1+0+0+1)*16+(0+0+1+1+1+1+0+0)*8+ \\
&\quad (1+0+1+0+1+0+1+0+1)*4+(1+0+1+0+1+0+1+0)*2+(1+0+1+0+1+0+1+0)= 252 \dots \text{c15}
\end{aligned}$$

$$\begin{aligned}
0+9+27+18+63+54+36+45 &= (0+0+0+0+1+1+1+1+1)*32+(0+0+1+1+1+1+0+0)*16+(0+1+1+0+1+0+0+1)*8+ \\
&\quad (0+0+0+0+1+1+1+1)*4+(0+0+1+1+1+1+0+0)*2+(0+1+1+0+1+0+0+1)= 252 \dots \text{pd1} \\
62+52+37+40+1+11+26+23 &= (1+1+1+1+0+0+0+0+0)*32+(1+1+0+0+0+0+1+1)*16+(1+0+0+1+0+1+1+0)*8+ \\
&\quad (1+1+1+0+0+0+0+0+1)*4+(1+0+0+0+0+1+1+1)*2+(0+0+1+0+1+1+0+1)= 252 \dots \text{pd2} \\
3+10+31+22+60+53+32+41 &= (0+0+0+0+0+1+1+1+1+1)*32+(0+0+1+1+1+1+0+0)*16+(0+1+1+0+1+0+0+1)*8+ \\
&\quad (0+0+1+1+1+1+0+0)*4+(1+1+1+1+0+0+0+0)*2+(1+0+1+0+0+1+0+1)= 252 \dots \text{pd3} \\
61+48+33+43+2+15+30+20 &= (1+1+1+1+0+0+0+0+0)*32+(1+1+0+0+0+0+1+1)*16+(1+0+0+1+0+1+1+0)*8+ \\
&\quad (1+0+0+0+0+1+1+1)*4+(0+0+0+1+1+1+1+0)*2+(1+0+1+1+0+1+0+0)= 252 \dots \text{pd4} \\
7+14+28+21+56+49+35+42 &= (0+0+0+0+1+1+1+1+1)*32+(0+0+1+1+1+1+0+0)*16+(0+1+1+0+1+0+0+1)*8+ \\
&\quad (1+1+1+1+0+0+0+0)*4+(1+1+0+0+0+0+1+1)*2+(1+0+0+1+0+1+1+0)= 252 \dots \text{pd5}
\end{aligned}$$

$$\begin{aligned}
7+10+27+17+56+53+36+46 &= (0+0+0+0+1+1+1+1+1)*32+(0+0+1+1+1+1+0+0)*16+(0+1+1+0+1+0+0+1)*8+ \\
&\quad (1+0+0+0+0+1+1+1)*4+(1+1+1+0+0+0+0+1)*2+(1+0+1+1+0+1+0+0)= 252 \dots \text{pb5} \\
57+48+37+44+6+15+26+19 &= (1+1+1+1+0+0+0+0+0)*32+(1+1+0+0+0+0+1+1)*16+(1+0+0+1+0+1+1+0)*8+ \\
&\quad (0+0+1+1+1+1+0+0)*4+(0+0+0+0+1+1+1+1)*2+(1+0+1+0+0+1+0+1)= 252 \dots \text{pb6} \\
4+14+31+18+59+49+32+45 &= (0+0+0+0+0+1+1+1+1+1)*32+(0+0+1+1+1+1+0+0)*16+(0+1+1+0+1+0+0+1)*8+ \\
&\quad (1+1+1+0+0+0+0+0+1)*4+(0+1+1+1+1+0+0+0)*2+(0+0+1+0+1+1+0+1)= 252 \dots \text{pb7} \\
58+51+33+40+5+12+30+23 &= (1+1+1+1+0+0+0+0+0)*32+(1+1+0+0+0+0+1+1)*16+(1+0+0+1+0+1+1+0)*8+ \\
&\quad (0+0+0+0+1+1+1+1)*4+(1+1+0+0+0+0+1+1)*2+(0+1+1+0+1+0+0+1)= 252 \dots \text{pb8}
\end{aligned}$$

... This is certainly a Pan-diagonal Magic Square!

$$\begin{aligned}
0+45=45 \dots \text{cp1}; 62+19=81 \dots \text{cp2}; 3+46=49 \dots \text{cp3}; 61+16=77 \dots \text{cp4}; \\
7+42=49 \dots \text{cp5}; 57+20=77 \dots \text{cp6}; 4+41=45 \dots \text{cp7}; 58+23=81 \dots \text{cp8}; \\
55+26=81 \dots \text{cp9}; 9+36=45 \dots \text{cp10}; 52+25=77 \dots \text{cp11}; 10+39=49 \dots \text{cp12}; \\
48+29=77 \dots \text{cp13}; 14+35=49 \dots \text{cp14}; 51+30=81 \dots \text{cp15}; 13+32=45 \dots \text{cp16}; \\
24+53=77 \dots \text{cp17}; 38+11=49 \dots \text{cp18}; 27+54=81 \dots \text{cp19}; 37+8=45 \dots \text{cp20}; \\
31+50=81 \dots \text{cp21}; 33+12=45 \dots \text{cp22}; 28+49=77 \dots \text{cp23}; 34+15=49 \dots \text{cp24}; \\
47+2=49 \dots \text{cp25}; 17+60=77 \dots \text{cp26}; 44+1=45 \dots \text{cp27}; 18+63=81 \dots \text{cp28}; \\
40+5=45 \dots \text{cp29}; 22+59=81 \dots \text{cp30}; 43+6=49 \dots \text{cp31}; 21+56=77 \dots \text{cp32};
\end{aligned}$$

As you see, both /D8i and /D4i prove to be useless, for you cannot always find CES of order 8 by them. But only /D2i proves to be useful for you to do it.

We have to modify a little about the Euler's Conditions for order 8 as follows:

*** New Euler's Conditions for Order 8 ***

- (1) Each line in each layer of /D2i forms must consist of {0, 0, 0, 0, 1, 1, 1, 1}.
Number '0' must be used as often as '1' in any line. One must not be used more or less often than the other, nor must be un-used in any line.
- (2) In the mathematical notation, any number of {0, 1, 2, 3, ..., 62, 63} must be used once. No number must be used twice or more often, nor must be unused.
- (3) Each line-sum must be expressed in the same form with the next equation:

$$V_{\text{sum}8} = (0+0+0+0+1+1+1+1)*32 + (0+0+0+0+1+1+1+1)*16 + (0+0+0+0+1+1+1+1)*8 + (0+0+0+0+1+1+1+1)*4 + (0+0+0+0+1+1+1+1)*2 + (0+0+0+0+1+1+1+1) = 252 \dots (E8)$$

You don't have to care about the permutation of {0, 0, 0, 0, 1, 1, 1, 1} here. You only have to care for the only constant combination.

If you accept these modifications, you can surely find many **Complete Euler Squares** of order 8 among various types of pan-diagonal MS88.

But how many Complete Euler Squares in all are there for order 8?

I know there are a lot of CES of order 8, but I have not yet succeeded in counting them through up to the last. I also know there are lots of "Non-Euler type" of pan-diagonal magic squares of order 8, say, far more than CES.

What types of pan-diagonal MS88 are all Complete Euler Squares, then?

Please take your kind look at the next list of 5 examples below.

**** Decomposition of Various Types of Pan-diagonal Magic Squares 8x8 ****

#1/Complete	/D2i
1 62 5 59 2 61 12 58	01010101 01010101 01010111 01100100 00010010 01001011
57 14 50 48 9 18 45 19	10110010 10100101 11011010 01010010 00010001 01110100
10 27 34 25 54 33 36 41	00101111 01011000 11010001 00001000 01000010 10101010
49 30 26 23 52 21 22 37	10001001 11111110 01100000 01010111 00011000 01101010
63 4 53 7 64 3 60 6	10101010 10101010 10001010 10111001 11011110 01001011
56 47 20 46 8 51 15 17	11010100 10100101 01010010 11011010 11101110 10111000
11 32 29 24 55 38 31 40	00001101 01111010 11100010 01111111 11011011 01010101
13 44 43 28 16 35 39 42	01100111 00010000 11111001 10001010 01111110 01011001
#2/Multiple, Complete	/D2i
1 63 4 62 15 49 34 32	01010110 01010101 01011001 01011001 01101001 00110011
60 7 54 9 41 39 20 30	10101100 10100011 10011001 01100101 11000110 10100011
10 52 21 47 36 23 28 43	01011001 01100110 10010011 00110100 01011111 11001010
59 8 51 12 38 19 48 25	10101010 10100101 10010011 01001010 11110110 01011010
50 16 31 33 64 2 61 3	10011010 10101010 01101010 01101010 01101001 11001100
24 26 45 35 5 58 11 56	00110101 11000101 01100110 10101001 10010011 11000101
29 42 37 22 55 13 44 18	01101010 10011001 11000110 10111100 00001010 01010011
27 46 17 40 6 57 14 53	01010101 10100101 11000110 01011011 10010000 01011010
#3/'C & P'	/D2i
1 64 3 62 7 58 5 60	01010101 01010101 01010101 01011010 01101001 01010101
63 2 61 4 57 8 59 6	10101010 10101010 10101010 10100101 10010110 01010101
25 40 27 38 31 34 29 36	01010101 10101010 10101010 01011010 01101001 01010101
39 26 37 28 33 32 35 30	10101010 01010101 01010101 10100101 10010110 01010101
50 15 52 13 56 9 54 11	10101010 10101010 01010101 01011010 01101001 10101010
16 49 14 51 10 55 12 53	01010101 01010101 10101010 10100101 10010110 10101010
42 23 44 21 48 17 46 19	10101010 01010101 10101010 01011010 01101001 10101010
24 41 22 43 18 47 20 45	01010101 10101010 01010101 10100101 10010110 10101010
#4/'C & C'	/D2i
1 63 5 59 14 52 10 56	01010101 01010101 01011010 01101001 01010101 00001111
62 4 58 8 49 15 53 11	10101010 10101010 10100101 10010110 01010101 11110000
17 47 21 43 30 36 26 40	01010101 10101010 01011010 01101001 01010101 00001111
46 20 42 24 33 31 37 27	10101010 01010101 10100101 10010110 01010101 11110000
51 13 55 9 64 2 60 6	10101010 10101010 01011010 01101001 10101010 00001111
16 50 12 54 3 61 7 57	01010101 01010101 10100101 10010110 10101010 11110000
35 29 39 25 48 18 44 22	10101010 01010101 01011010 01101001 10101010 00001111
32 34 28 38 19 45 23 41	01010101 10101010 10100101 10010110 10101010 11110000

```
#5/Multiple, 'C&C'           /D2i
  1 63  6 60 38 28 33 31   01011010 01010101 01010101 01101001 01010101 00111100
 62  4 57  7 25 39 30 36   10100101 10101010 10101010 10010110 01010101 11000011
11 53 16 50 48 18 43 21   01011010 01010101 10101010 01101001 10101010 00111100
56 10 51 13 19 45 24 42   10100101 10101010 01010101 10010110 10101010 11000011
27 37 32 34 64  2 59  5   01011010 10101010 10101010 01101001 10101010 00111100
40 26 35 29  3 61  8 58   10100101 01010101 01010101 10010110 10101010 11000011
17 47 22 44 54 12 49 15   01011010 10101010 01010101 01101001 01010101 00111100
46 20 41 23  9 55 14 52   10100101 01010101 10101010 10010110 01010101 11000011
```

The first two examples of 'Complete' MS88 are not Complete Euler Squares. You can find any other patterns than {0, 0, 0, 0, 1, 1, 1, 1} here and there in each /D2i layer.

The last three solutions above have no other patterns than {0, 0, 0, 0, 1, 1, 1, 1} in any line in each layer, even in all pan-diagonals, as you see.

It seems 'Composite' type of pan-diagonal MS88 are all Complete Euler Squares, whatever they might be even 'Complete', 'Multiple' and 'Simultaneous' types at the same time.

#9. How about the Magic Squares of Order 9?

I would like you to see the next list of special example below. It is 'multiple' type of Simultaneous MS99: Self-complementary and Pan-diagonal.

If you watch the /D9i diagrams and examine each number pattern in any line in any layer, you will easily know this is not a Complete Euler Square at all.

** 'Simultaneous' Magic Squares 9x9 with /D9i **

```
1/Classic           /Mathemat  .-.-.-.-.-.
  1 42 80 65 22 36 48 59 16   35 47 58 15| 0|41|79|64|21|35|47|58|15| 0 41 79 64
50 61 12  6 44 73 67 27 29   72 66 26 28|49|60|11| 5|43|72|66|26|28|49 60 11  5
72 20 31 52 57 14  8 37 78   13  7 36 77|71|19|30|51|56|13| 7|36|77|71 19 30 51
 7 39 77 71 19 33 54 56 13   32 53 55 12| 6|38|76|70|18|32|53|55|12| 6 38 76 70
47 58 18  3 41 79 64 24 35   78 63 23 34|46|57|17| 2|40|78|63|23|34|46 57 17  2
69 26 28 49 63 11  5 43 75   10  4 42 74|68|25|27|48|62|10| 4|42|74|68 25 27 48
 4 45 74 68 25 30 51 62 10   29 50 61  9| 3|44|73|67|24|29|50|61| 9| 3 44 73 67
53 55 15  9 38 76 70 21 32   75 69 20 31|52|54|14| 8|37|75|69|20|31|52 54 14  8
66 23 34 46 60 17  2 40 81   16  1 39 80|65|22|33|45|59|16| 1|39|80|65 22 33 45
```

```
/N9i           /D9i           *9/           *1/
00 45 87 71 23 38 52 64 16   0 4 8 7 2 3 5 6 1   0 5 7 1 3 8 2 4 6
54 66 12 05 47 80 73 28 31   5 6 1 0 4 8 7 2 3   4 6 2 5 7 0 3 8 1
78 21 33 56 62 14 07 40 85   7 2 3 5 6 1 0 4 8   8 1 3 6 2 4 7 0 5
06 42 84 77 20 35 58 61 13   0 4 8 7 2 3 5 6 1   6 2 4 7 0 5 8 1 3
51 63 18 02 44 86 70 25 37   5 6 1 0 4 8 7 2 3   1 3 8 2 4 6 0 5 7
75 27 30 53 68 11 04 46 82   7 2 3 5 6 1 0 4 8   5 7 0 3 8 1 4 6 2
03 48 81 74 26 32 55 67 10   0 4 8 7 2 3 5 6 1   3 8 1 4 6 2 5 7 0
57 60 15 08 41 83 76 22 34   5 6 1 0 4 8 7 2 3   7 0 5 8 1 3 6 2 4
72 24 36 50 65 17 01 43 88   7 2 3 5 6 1 0 4 8   2 4 6 0 5 7 1 3 8
```

* Check Line-Sums *

```
0+41+79+64+21+35+47+58+15= (0+4+8+7+2+3+5+6+1)*9+(0+5+7+1+3+8+2+4+6)= 360 ... rw1
49+60+11+5+43+72+66+26+28= (5+6+1+0+4+8+7+2+3)*9+(4+6+2+5+7+0+3+8+1)= 360 ... rw2
71+19+30+51+56+13+7+36+77= (7+2+3+5+6+1+0+4+8)*9+(8+1+3+6+2+4+7+0+5)= 360 ... rw3
6+38+76+70+18+32+53+55+12= (0+4+8+7+2+3+5+6+1)*9+(6+2+4+7+0+5+8+1+3)= 360 ... rw4

0+49+71+6+46+68+3+52+65= (0+5+7+0+5+7+0+5+7)*9+(0+4+8+6+1+5+3+7+2)= 360 ... c11
41+60+19+38+57+25+44+54+22= (4+6+2+4+6+2+4+6+2)*9+(5+6+1+2+3+7+8+0+4)= 360 ... c12
79+11+30+76+17+27+73+14+33= (8+1+3+8+1+3+8+1+3)*9+(7+2+3+4+8+0+1+5+6)= 360 ... c13
```


$$64+5+51+70+2+48+67+8+45 = (7+0+5+7+0+5+7+0+5) * 9 + (1+5+6+7+2+3+4+8+0) = 360 \dots c14$$

$$0+60+30+70+40+10+50+20+80 = (0+6+3+7+4+1+5+2+8) * 9 + (0+6+3+7+4+1+5+2+8) = 360 \dots pd1$$

$$41+11+51+18+78+4+61+31+65 = (4+1+5+2+8+0+6+3+7) * 9 + (5+2+6+0+6+4+7+4+2) = 360 \dots pd2$$

$$79+5+56+32+63+42+9+52+22 = (8+0+6+3+7+4+1+5+2) * 9 + (7+5+2+5+0+6+0+7+4) = 360 \dots pd3$$

$$64+43+13+53+23+74+3+54+33 = (7+4+1+5+2+8+0+6+3) * 9 + (1+7+4+8+5+2+3+0+6) = 360 \dots pd4$$

$$35+43+51+76+57+68+9+20+1 = (3+4+5+8+6+7+1+2+0) * 9 + (8+7+6+4+3+5+0+2+1) = 360 \dots pb6$$

$$47+72+56+70+17+25+3+31+39 = (5+8+6+7+1+2+0+3+4) * 9 + (2+0+2+7+8+7+3+4+3) = 360 \dots pb7$$

$$58+66+13+18+2+27+44+52+80 = (6+7+1+2+0+3+4+5+8) * 9 + (4+3+4+0+2+0+8+7+8) = 360 \dots pb8$$

$$15+26+7+32+40+48+73+54+65 = (1+2+0+3+4+5+8+6+7) * 9 + (6+8+7+5+4+3+1+0+2) = 360 \dots pb9$$

**** The same 'Simultaneous' Magic Squares 9x9 with /D3i ****

/N3i		/D3i	*27/	*9/	*3/	*1/
0000	1112 2221 2101 0210 1022 1202 2011 0120	012201120	012120201	012012012	021102210	
1211	2020 0102 0012 1121 2200 2110 0222 1001	120012201	201012120	120120120	102210021	
2122	0201 1010 1220 2002 0111 0021 1100 2212	201120012	120201012	201201201	210021102	
0020	1102 2211 2121 0200 1012 1222 2001 0110	012201120	012120201	201201201	021102210	
1201	2010 0122 0002 1111 2220 2100 0212 1021	120012201	201012120	012012012	102210021	
2112	0221 1000 1210 2022 0101 0011 1120 2202	201120012	120201012	120120120	210021102	
0010	1122 2201 2111 0220 1002 1212 2021 0100	012201120	012120201	120120120	021102210	
1221	2000 0112 0022 1101 2210 2120 0202 1011	120012201	201012120	201201201	102210021	
2102	0211 1020 1200 2012 0121 0001 1110 2222	201120012	120201012	012012012	210021102	

*** Check Line-Sums ***

$$0+41+79+64+21+35+47+58+15 = (0+1+2+2+0+1+1+2+0) * 27 + (0+1+2+1+2+0+2+0+1) * 9 + (0+1+2+0+1+2+0+1+2) * 3 + (0+2+1+1+0+2+2+1+0) = 360 \dots rw1$$

$$49+60+11+5+43+72+66+26+28 = (1+2+0+0+1+2+2+0+1) * 27 + (2+0+1+0+1+2+1+2+0) * 9 + (1+2+0+1+2+0+1+2+0) * 3 + (1+0+2+2+1+0+0+2+1) = 360 \dots rw2$$

$$71+19+30+51+56+13+7+36+77 = (2+0+1+1+2+2+0+0+1+2) * 27 + (1+2+0+2+0+1+0+1+2) * 9 + (2+0+1+2+0+1+2+0+1) * 3 + (2+1+0+0+2+1+1+0+2) = 360 \dots rw3$$

$$6+38+76+70+18+32+53+55+12 = (0+1+2+2+0+1+1+2+0) * 27 + (0+1+2+1+2+0+2+0+1) * 9 + (2+0+1+2+0+1+2+0+1) * 3 + (0+2+1+1+0+2+2+1+0) = 360 \dots rw4$$

$$0+49+71+6+46+68+3+52+65 = (0+1+2+0+1+2+0+1+2) * 27 + (0+2+1+0+2+1+0+2+1) * 9 + (0+1+2+2+0+1+1+2+0) * 3 + (0+1+2+0+1+2+0+1+2) = 360 \dots c11$$

$$41+60+19+38+57+25+44+54+22 = (1+2+0+1+2+0+1+2+0) * 27 + (1+0+2+1+0+2+1+0+2) * 9 + (1+2+0+0+1+2+2+0+1) * 3 + (2+0+1+2+0+1+2+0+1) = 360 \dots c12$$

$$79+11+30+76+17+27+73+14+33 = (2+0+1+2+0+1+2+0+1) * 27 + (2+1+0+2+1+0+2+1+0) * 9 + (2+0+1+1+2+0+0+1+2) * 3 + (1+2+0+1+2+0+1+2+0) = 360 \dots c13$$

$$64+5+51+70+2+48+67+8+45 = (2+0+1+2+0+1+2+0+1) * 27 + (1+0+2+1+0+2+1+0+2) * 9 + (0+1+2+2+0+1+1+2+0) * 3 + (1+2+0+1+2+0+1+2+0) = 360 \dots c14$$

$$0+60+30+70+40+10+50+20+80 = (0+2+1+2+1+0+1+0+2) * 27 + (0+0+0+1+1+1+2+2+2) * 9 + (0+2+1+2+1+0+1+0+2) * 3 + (0+0+0+1+1+1+2+2+2) = 360 \dots pd1$$

$$41+11+51+18+78+4+61+31+65 = (1+0+1+0+2+0+2+1+2) * 27 + (1+1+2+2+2+0+0+0+1) * 9 + (1+0+2+0+2+1+2+1+0) * 3 + (2+2+0+0+0+1+1+1+2) = 360 \dots pd2$$

$$79+5+56+32+63+42+9+52+22 = (2+0+2+1+2+1+0+1+0) * 27 + (2+0+0+0+1+1+1+2+2) * 9 + (2+1+0+1+0+2+0+2+1) * 3 + (1+2+2+2+0+0+0+1+1) = 360 \dots pd3$$

$$64+43+13+53+23+74+3+54+33 = (2+1+0+1+0+2+0+2+1) * 27 + (1+1+1+2+2+2+0+0+0) * 9 + (0+2+1+2+1+0+1+0+2) * 3 + (1+1+1+2+2+2+0+0+0) = 360 \dots pd4$$

$$35+43+51+76+57+68+9+20+1 = (1+1+1+2+2+2+0+0+0) * 27 + (0+1+2+2+0+1+1+2+0) * 9 + (2+2+2+1+1+1+0+0+0) * 3 + (2+1+0+1+0+2+0+2+1) = 360 \dots pb6$$

$$47+72+56+70+17+25+3+31+39 = (1+2+2+2+0+0+0+1+1) * 27 + (2+2+0+1+1+2+0+0+1) * 9 + (0+0+0+2+2+2+1+1+1) * 3 + (2+0+2+1+2+1+0+1+0) = 360 \dots pb7$$

$$58+66+13+18+2+27+44+52+80 = (2+2+0+0+0+1+1+1+2) * 27 + (0+1+1+2+0+0+1+2+2) * 9 + (1+1+1+0+0+0+2+2+2) * 3 + (1+0+1+0+2+0+2+1+2) = 360 \dots pb8$$

$$15+26+7+32+40+48+73+54+65 = (0+0+0+1+1+1+2+2+2) * 27 + (1+2+0+0+1+2+2+0+1) * 9 + (2+2+2+1+1+1+0+0+0) * 3 + (0+2+1+2+1+0+1+0+2) = 360 \dots pb9$$

... This is certainly a Pan-diagonal Magic Square!

$$0+80=80 \dots cp1; 41+39=80 \dots cp2; 79+1=80 \dots cp3; 64+16=80 \dots cp4;$$

$$21+59=80 \dots cp5; 35+45=80 \dots cp6; 47+33=80 \dots cp7; 58+22=80 \dots cp8;$$

$$15+65=80 \dots cp9; 49+31=80 \dots cp10; 60+20=80 \dots cp11; 11+69=80 \dots cp12;$$

$$5+75=80 \dots cp13; 43+37=80 \dots cp14; 72+8=80 \dots cp15; 66+14=80 \dots cp16;$$

26+54=80 ... cp17; 28+52=80 ... cp18; 71+9=80 ... cp19; 19+61=80 ... cp20;
 30+50=80 ... cp21; 51+29=80 ... cp22; 56+24=80 ... cp23; 13+67=80 ... cp24;
 7+73=80 ... cp25; 36+44=80 ... cp26; 77+3=80 ... cp27; 6+74=80 ... cp28;
 38+42=80 ... cp29; 76+4=80 ... cp30; 70+10=80 ... cp31; 18+62=80 ... cp32;
 32+48=80 ... cp33; 53+27=80 ... cp34; 55+25=80 ... cp35; 12+68=80 ... cp36;
 46+34=80 ... cp37; 57+23=80 ... cp38; 17+63=80 ... cp39; 2+78=80 ... cp40;
 40+40=80 ... cp41;

... This is certainly a Self-complementary type!

But if you watch the /D3i diagrams of the second list of the same solution above and examine each number pattern in any line in any layer, you will surely find it is certainly a Complete Euler Square. It is the decompositions by the base 3 that you must have and draw the diagrams by. The more CES if you want to find, the better base of decompositions you should have.

Let's modify a little about the Euler's Conditions for order 9 as follows:

*** New Euler's Conditions for Order 9 ***

(1) Each line in each layer of /D3i forms must consist of {0, 0, 0, 1, 1, 1, 2, 2, 2}.

Any number of {0, 1 and 2} must be used as often as the others. One must not be used more or less often than the others, nor must be un-used in any line.

(2) In the mathematical notation, any number of {0, 1, 2, 3, ..., 79, 80} must be used once. No number must be used twice or more often, nor must be unused.

(3) Each line-sum must be expressed in the same form with the next equation:

$$V_{sum9} = (0+0+0+1+1+1+2+2+2) * 27 + (0+0+0+1+1+1+2+2+2) * 9 + (0+0+0+1+1+1+2+2+2) * 3 + (0+0+0+1+1+1+2+2+2) = 360 \dots (E9)$$

You don't have to care about the permutation of {0, 0, 0, 1, 1, 1, 2, 2, 2} here. You only have to care for the only constant combination.

If you accept these modifications, you can surely find many Complete Euler Squares of order 9 among various types of pan-diagonal MS99.

But how many Complete Euler Squares of order 9 are there in all?

I know there are not so many C.E.S. of order 9 as uncountable, but I have not yet counted them through. I am afraid it may take a very long time to do that.

To tell the truth, I had always failed in finding CES of order 9 for a long time. I had wondered if there might be no CES of order 9, just before I found some real samples of CES among the multiple type of pan-diagonal ones including 9 little squares of order 3 within. I could find only a small number of solutions of that type, then.

It is getting really hard to find any CES of higher orders than 6. The higher orders, the fewer CES you can find. Those of higher orders are indeed rare, precious jewels.

I know nothing about how many "Non-Euler type" of MS99 exist. I am not interested in counting them through, because it seems to be practically impossible.

#10. Comment

What does the existence of 'Complete Euler Squares' mean?

I know it is a difficult question for us to answer, but the Complete Euler Squares seemed to be the common, essential structures of every pan-diagonal magic squares of lower orders. They also seemed to mean the fundamental structures of all pan-diagonal magic squares of higher orders. But, were those impressions really true?

They really seemed to be true just as Legendary L. Euler had wanted them to be.

But another idea came up to me and has been growing up these days.

Complete Euler Squares may not always mean the essential structure of all pan-diagonal magic squares in general, but they may really mean the special key frames of composing the rare, precious jewels of special kinds.

All pan-diagonal MS44 and MS55 are certainly Complete Euler Squares. But it is also true they are those rare, precious jewels of special kinds at the same time.

Miracle might have existed all the time from the beginning of magic squares.

But, I should only say now that we have to study more about them and discuss precisely with a lot of data about what they essentially mean in general.

At the end of this section, I want to tell you about my deepest appreciation to Dr. Mutsumi Suzuki. He has given me a lot of advice and inspiration to study magic squares with these modern mathematical arts, since I had never been fond of new notations and made myself stick to the classical style only.

"Without having any knowledge of those, you might surely miss any of the most important things. You will not be able to have the master keys to open the doors. Study 'Latin Squares', and learn some new notations at first," he once told me.

Now I would like to send this same message to you all beginners.

(Written on Oct.29, 2005 by Kanji Setsuda with MacOSX and Xcode;
Retyped on Feb.20, 2015 with MacOSX 10.10.2 and Xcode 6.1.1;)

E-mail Address:<jag12001@nifty.com>